Conjugate prior in the Mallows model with Spearman distance

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Introduction

- **Ranking and comparing items:** crucial for collecting information about preferences in many areas: e.g. marketing, business, politics, genetics.
- **Mallows model:** powerful model for rankings
- **Question:** How to include **expert opinions** into the analysis? How to elicit a meaningful **prior on the consensus** ranking of a population?

1. The Mallows model with Spearman distance

The **Mallows model** (Mallows 1957) is a class of non-uniform distributions for $\boldsymbol{R} \in \mathcal{P}_n$, the set of n-dimensional permutations, of the form

$$P(\boldsymbol{R}|\theta,\boldsymbol{\rho}) = \frac{e^{-\theta d(\boldsymbol{R},\boldsymbol{\rho})}}{Z(\theta)}$$

- $\boldsymbol{\rho} \in \mathcal{P}_n$: consensus ranking, location parameter
- $\theta \geq 0$: precision parameter
- $d(\cdot,\cdot)$: right-invariant (Diaconis 1988) distance
- $Z(\theta) = \sum_{\boldsymbol{r}\in\mathcal{P}_n} e^{-\theta d(\boldsymbol{r},\boldsymbol{1}_n)}$: partition function (independent of $\boldsymbol{\rho}$ because of right-invariance of $d(\cdot,\cdot)$)

Spearman distance: the squared l_2 norm on $\mathcal{P}_n$

$$d_S(\boldsymbol{\rho},\boldsymbol{\sigma}) = \sum_{i=1}^{n}(\rho_i-\sigma_i)^2, \quad \boldsymbol{\rho},\boldsymbol{\sigma}\in\mathcal{P}_n$$

Consider n items, ranked by N assessors.
Denote by $\mathbf{R}_j = (R_{1j}, R_{2j}, \ldots, R_{nj}) \in \mathcal{P}_n$, the ranking of user j, $j = 1, ..., N$.

Under the Mallows model with Spearman distance (MMS), given θ, the likelihood can be written as

$$P(\boldsymbol{R}_1,...,\boldsymbol{R}_N|\boldsymbol{\rho}) \propto \exp\left(2\theta N\sum_{i=1}^{n}\rho_i\bar{R}_i\right)$$

where $\bar{R}_i = \frac{1}{N}\sum_{j=1}^{N} R_{ij}$, $i = 1,...,n$, is the sample average of the i-th rank.

2. Sufficient statistics and mle

The sufficient statistic for $\boldsymbol{\rho} = (\rho_1,...,\rho_n)$, when θ is known, is $\bar{\boldsymbol{R}} = (\bar{R}_1,...,\bar{R}_N)$.

Proposition 1. *Let $\boldsymbol{R}_1,...,\boldsymbol{R}_N|\boldsymbol{\rho},\theta \sim Mall(\theta,\boldsymbol{\rho})$, and define the vector of sample ranks $\bar{\boldsymbol{R}}$ as above. Assume $\bar{R}_i \neq \bar{R}_j$, for each $i \neq j$, and denote by $\boldsymbol{Y}(\bar{\boldsymbol{R}}) = [Y_1(\bar{\boldsymbol{R}}),\cdots,Y_n(\bar{\boldsymbol{R}})] \in \mathcal{P}_n$ the rank vector of $\bar{\boldsymbol{R}}$, i.e. $Y_i(\bar{\boldsymbol{R}}) = Y_i = \sum_{h=1}^{n}\mathbb{1}(\bar{R}_h \leq \bar{R}_i)$, $i = 1,...,n$. Then the unique mle of $\boldsymbol{\rho}$ is*

$$\boldsymbol{\rho}_{mle} = \underset{\boldsymbol{\rho}\in\mathcal{P}_n}{\operatorname{argmax}}\sum_{i=1}^{n}\rho_i\bar{R}_i = \boldsymbol{Y}(\bar{\boldsymbol{R}}).$$

Notice that, in general, $\bar{\boldsymbol{R}} \notin \mathcal{P}_n$. However $\bar{\boldsymbol{R}}$ lives in the permutohedron of order n.

Definition 1. *The **permutohedron** of order n, $\mathbb{PP}_n$, is an $(n-1)$-dimensional polytope embedded in an n-dimensional space, the vertices of which are formed by permuting the coordinates of the vector $(1,2,3,...,n)$. Equivalently, it is the convex hull of the points $\boldsymbol{\rho} \in \mathcal{P}_n$, the set of n-dim permutations.*

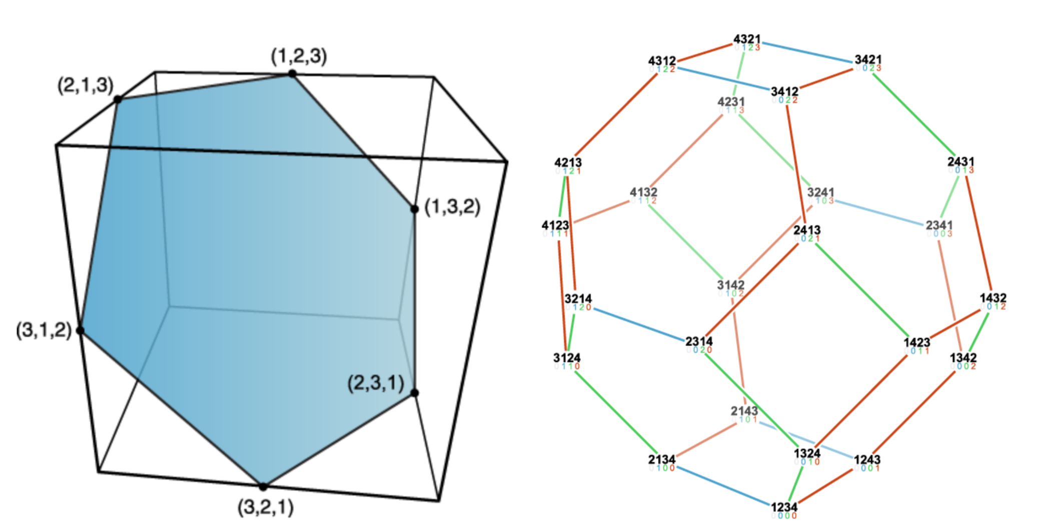

Figure 1: Left: The permutohedron of order 3 is a regular hexagon, filling a cross-section of a $2\times2\times2$ cube; Right: The permutohedron of order 4 is a truncated octahedron.

3. The Bayesian Mallows

Vitelli et al. (2018) proposed a framework for performing **Bayesian inference on the Mallows** model.

They assume that $\boldsymbol{\rho}$ is a priori uniformly distributed, $\boldsymbol{\rho} \sim \text{Unif}(\mathcal{P}_n)$.

Contribution: the objective prior in the sense of Villa & Walker (2015) is the uniform prior density on the space of permutations $\boldsymbol{\rho} \sim \text{Unif}(\mathcal{P}_n)$.

Can we go further?
How to put an informative prior on $\boldsymbol{\rho}$?

A: θ given: conjugate prior for $\boldsymbol{\rho}$

B: θ not given: conjugate conditioned on θ + clever prior on θ $\rightarrow$ MH sampling scheme to approximate the posterior

3.A Conjugate prior for $\boldsymbol{\rho}$ (given θ)

Proposition 2. *Keeping θ fixed, **the conjugate prior** for $\rho \in \mathcal{P}_n$ is*

$$\pi(\boldsymbol{\rho}|\boldsymbol{\rho}_0,\theta N_0) = \frac{\exp\left[-\theta N_0\sum_{i=1}^{n}(\rho_i-\rho_{0i})^2\right]}{Z^*(\theta N_0,\rho_0)},$$

where $\boldsymbol{\rho}_0 \in \mathbb{PP}_n$, and $N_0 \in \mathbb{N}$. We call $\pi(\cdot|\boldsymbol{\rho}_0,\theta N_0)$ the extended Mallows density: it is a Mallows model where the consensus $\boldsymbol{\rho}_0$ belongs to $\mathbb{PP}_n$.

Posterior consensus: weighted average of prior parameter and observed mean (recall Diaconis et al. 1979):

$$\boldsymbol{\rho}_N = \frac{N}{N_0+N}\bar{\boldsymbol{R}} + \frac{N_0}{N_0+N}\boldsymbol{\rho}_0 \in \mathbb{PP}_n$$
$$\theta_N = \theta(N_0+N) \in [0,\infty)$$

Then N_0 can be interpreted as an equivalent sample size (recall the Gaussian).

Remark 1. *When $N_0 = 0$, the conjugate prior reduces to the uniform, for all $\boldsymbol{\rho}_0$. When $\boldsymbol{\rho}_0 = \left(\frac{n+1}{2},...,\frac{n+1}{2}\right)$ the conjugate prior reduces to the uniform, for all θ_0.*

3.A.bis Toy example 1

Sample $N = 30$ rankings from the MMS with $\boldsymbol{\rho} = (3,1,2)$ and $\theta = 0.18$. Conjugate prior with $\boldsymbol{\rho}_0 = (1,2,3)$, and varying $\theta_0 = \theta N_0$.

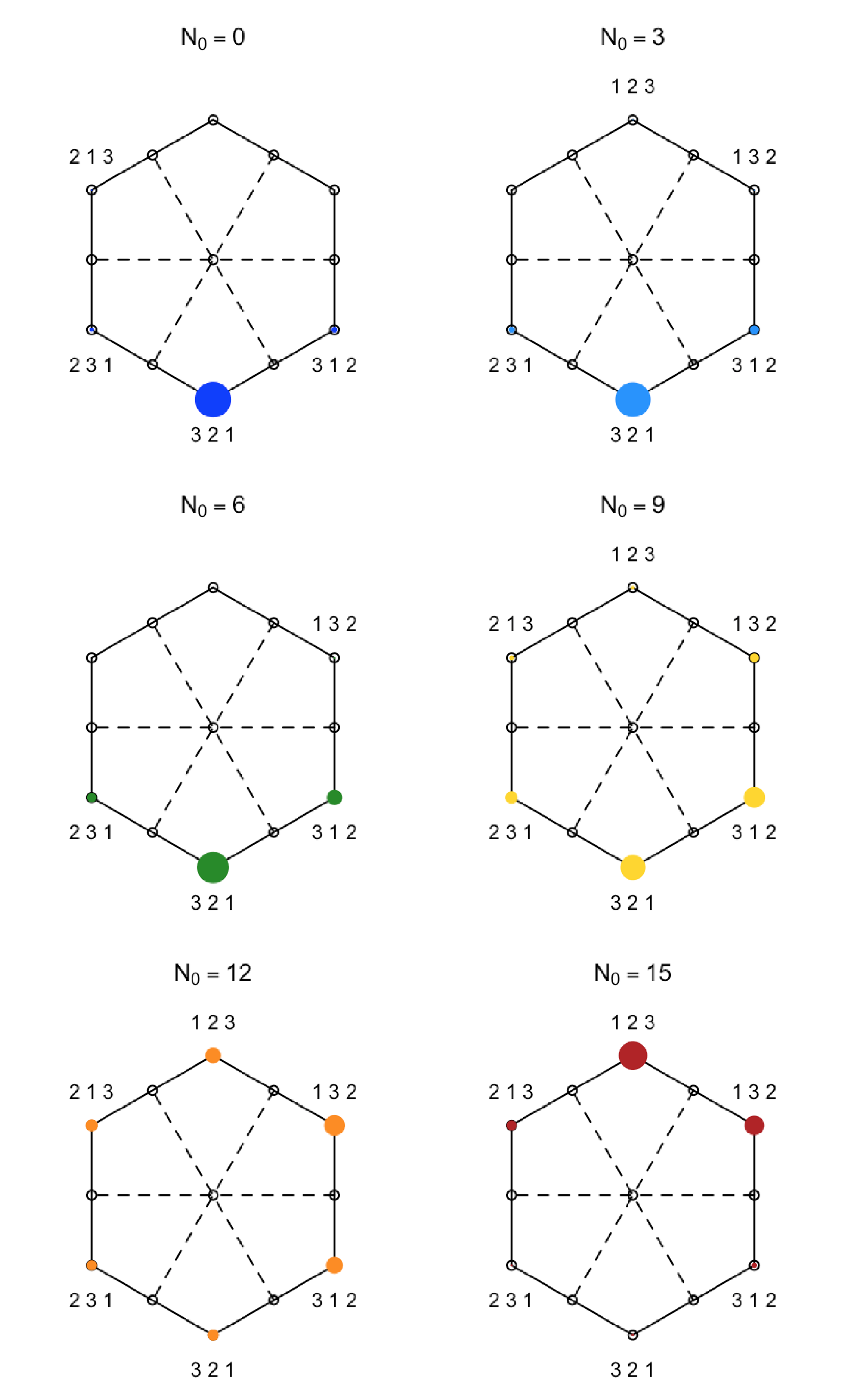

Figure 2: The balls have radius proportional to the frequency of rankings in the posterior sample.

3.A.bis Toy example 2

Same model. Increasing sample size (N). Conjugate prior with $\boldsymbol{\rho}_0 = (1,2,3)$ and $\theta N_0 = 3$ (i.e. $N_0 \approx 16$).

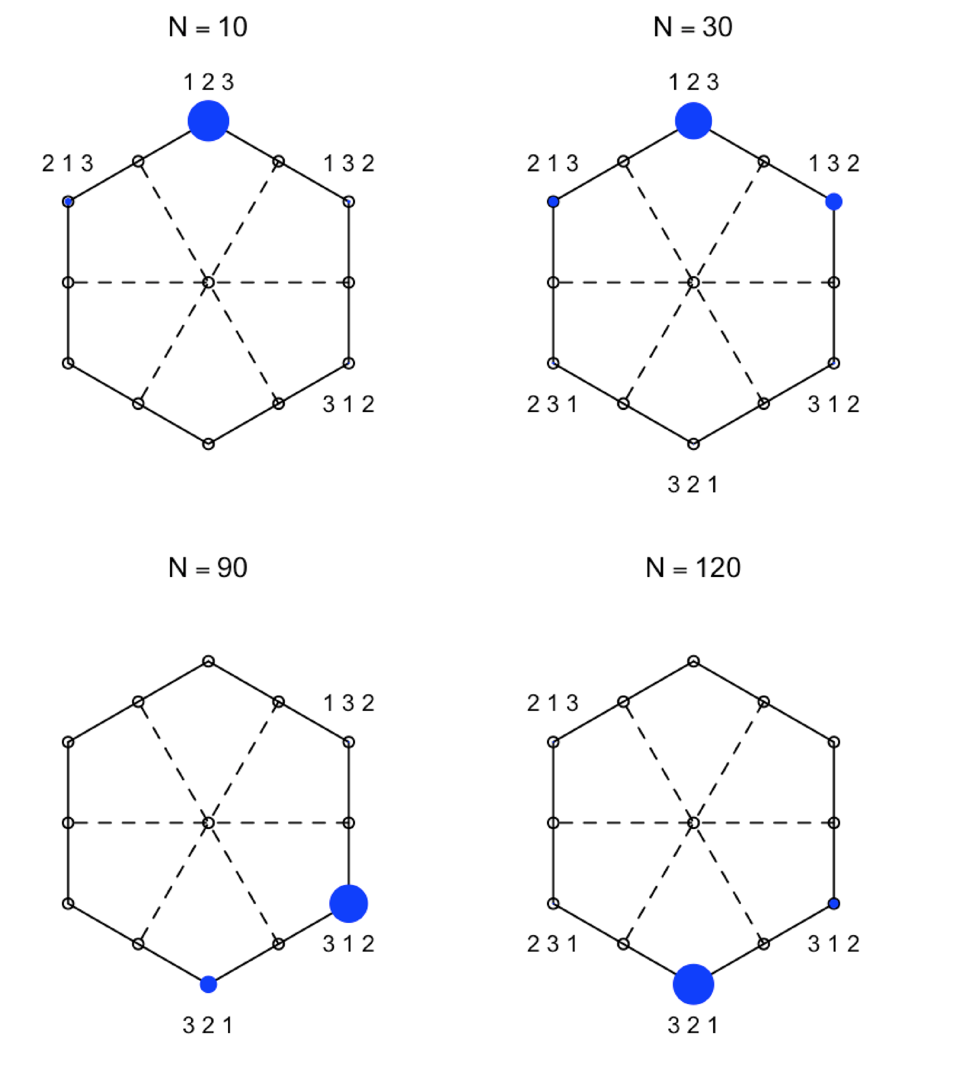

Figure 3: The balls have radius proportional to the frequency of rankings in the posterior sample.

3.B Prior when θ not given

When θ is unknown, the partition function of $\pi(\cdot|\boldsymbol{\rho}_0,\theta N_0)$ depends on the model parameters and cannot be avoided.

Solution: Let $\pi(\theta) \propto Z^*(\theta N_0,\boldsymbol{\rho}_0)$, so that the posterior full conditional is treatable.

3.B Prior when θ not given

Same model. Increasing sample size (N). $\boldsymbol{\rho}_0 = (1,2,3)$ and $N_0 = 16$.

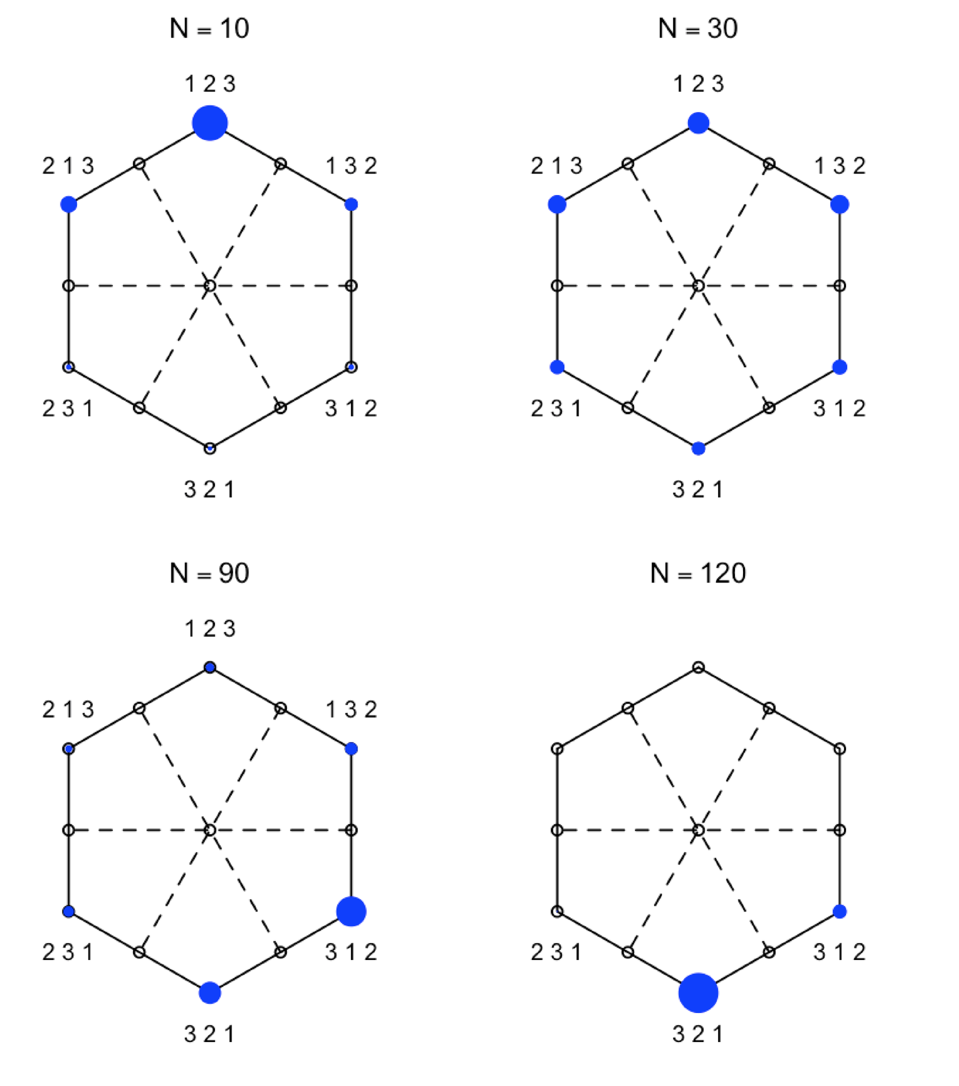

Figure 4: The balls have radius proportional to the frequency of rankings in the posterior sample.

Ongoing work

- Can we say something about the convergence rate?
  $\rightarrow$ Can we say something about $Z^*(\cdot,\cdot)$?
- Applications?

Interesting data to test our methods on? Please take contact!

References

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